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Functional Relations and Bethe Ansatz for the XXZ Chain

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There is an approach due to Bazhanov and Reshetikhin for solving integrable RSOS models which consists of solving the functional relations which result from the truncation of the fusion hierarchy. We demonstrate that this is also an effective means of solving integrable vertex models. Indeed, we use this method to recover the known Bethe Ansatz solutions of both the closed and open XXZ quantum spin chains with $U(1)$ symmetry. Moreover, since this method does not rely on the existence of a pseudovacuum state, we also use this method to solve a special case of the open XXZ chain with nondiagonal boundary terms.

KEY WORDS: Functional relations; fusion hierarchy; Bethe Ansatz; Yang–Baxter equation; integrable quantum spin chain; boundary terms; boundary sine-Gordon model.

1. INTRODUCTION

There are several well-known methods of deriving the Bethe Ansatz (BA) solution of integrable vertex models: the coordinate BA,[(1–3)] the $T-Q$ approach,[(2)] the algebraic BA,[(4–6)] the analytic BA,[(7)] and the functional BA.[(8)] We present here yet another method, which entails solving the functional relations which result from the truncation of a model's fusion hierarchy. This approach was (to our knowledge) first developed for RSOS models[(9)] by Bazhanov and Reshetikhin in ref. 10, but until now has not been applied to vertex-type models. An important feature of this method is that, unlike some of the other approaches, it does not rely on the existence of a pseudovacuum (reference) state.

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Our primary motivation for this work comes from the long outstanding problem of solving the open spin-$\frac{1}{2}$ XXZ quantum spin chain with nondiagonal boundary terms, defined by the Hamiltonian[11]

$$\mathcal{H} = \frac{1}{2}\left\{\sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh\eta\, \sigma_n^z \sigma_{n+1}^z) \right.$$
$$\left. + \sinh\eta \left(\coth\xi_- \sigma_1^z + \frac{2\kappa_-}{\sinh\xi_-}\sigma_1^x - \coth\xi_+ \sigma_N^z - \frac{2\kappa_+}{\sinh\xi_+}\sigma_N^x \right)\right\}, \quad (1.1)$$

where $\sigma^x, \sigma^y, \sigma^z$ are the standard Pauli matrices, η is the bulk anisotropy parameter, $\xi_\pm, \kappa_\pm$ are arbitrary boundary parameters, and N is the number of spins. This model is integrable. Indeed, the Hamiltonian is obtained from the commuting transfer matrix[6] constructed with the nondiagonal K matrix found in refs. 11 and 12 together with the standard XXZ R matrix.

Solving this problem (e.g., determining the energy eigenvalues in terms of roots of a system of Bethe Ansatz equations) is a crucial step in formulating the thermodynamics of the spin chain and of the boundary sine-Gordon model.[12] Moreover, this problem has important applications in condensed matter physics and statistical mechanics.

A fundamental difficulty is that, in contrast to the special case of diagonal boundary terms (i.e., $\kappa_\pm = 0$, in which case $\mathcal{H}$ has a $U(1)$ symmetry) considered in refs. 3 and 6, a simple pseudovacuum state does *not* exist (e.g., the state with all spins up is not an eigenstate of the Hamiltonian).

Some progress on this problem was made recently in refs. 13 and 14. Namely, for bulk anisotropy value $\eta = \frac{i\pi}{p+1}$, $p = 1, 2, \ldots$, (and hence $q \equiv e^\eta$ is a root of unity, satisfying $q^{p+1} = -1$), an exact $(p+1)$-order functional relation for the fundamental transfer matrix was proposed. The key observation is that the fused spin-$\frac{p+1}{2}$ transfer matrix can be expressed in terms of a lower-spin transfer matrix, resulting in the truncation of the fusion hierarchy.[2] The simplest case $p = 1$, which corresponds to the XX chain, is analyzed in ref. 13.

Although sets of equations for the eigenvalues of the transfer matrix were found in ref. 14, these equations do not have the standard Bethe Ansatz form, and they become increasingly complicated as the value of p increases. Moreover, one would like to solve the model for general values of η, i.e., not just for the discrete values corresponding to roots of unity.

We also achieve here some progress on these questions. In particular, from the functional relations in ref. 14, we obtain standard Bethe Ansatz

[2] This is distinct from the observation due to Belavin *et al.*[15,16] that, for the special case of quantum-group symmetry (i.e., $\kappa_\pm = 0$, $\xi_\pm \to \infty$), the fused transfer matrix $t^{(\frac{p}{2})}(u)$ vanishes after quantum group reduction.

equations for the transfer matrix eigenvalues for general values of η, albeit only for the special case

$$\kappa_+ = \kappa_- \equiv \kappa \neq 0, \qquad \xi_+ = \xi_- \equiv \xi, \qquad N = \text{odd}. \tag{1.2}$$

Unfortunately, we have not yet succeeded to obtain corresponding results for general values of the boundary parameters.

The outline of this article is as follows. In Section 2, we consider as a warm-up the case of the closed XXZ chain, and provide a new derivation of the well-known Bethe Ansatz solution. In Section 3, we turn to the open XXZ chain. Using the functional relations proposed in ref. 14, we first recover the solution of Alcaraz *et al.*[3] and Sklyanin[6] for the diagonal case $\kappa_\pm = 0$, and we then give the solution for the nondiagonal case (1.2). We conclude with a brief discussion of our results in Section 4.

2. THE CLOSED CHAIN

The closed (i.e., with periodic boundary conditions) spin-$\frac{1}{2}$ XXZ quantum spin chain is defined by the Hamiltonian

$$H = \frac{1}{2}\sum_{n=1}^{N} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh\eta\, \sigma_n^z \sigma_{n+1}^z), \tag{2.1}$$

where $\vec{\sigma}_{N+1} = \vec{\sigma}_1$. As noted in the Introduction, there are various methods of deriving the Bethe Ansatz solution of this model. As a warm-up for the open-chain problem, we now give another derivation of this solution, which involves solving the model's functional relations. In Section 2.1 we derive the functional relations, and then in Section 2.2 we proceed to solve them.

2.1. Functional Relations

In this subsection, we begin by briefly reviewing the construction of the (fused) transfer matrices of the closed XXZ chain. We then recall the so-called fusion hierarchy which these transfer matrices obey. Finally, we give an identity which truncates the fusion hierarchy and leads to the desired functional relations.

The fundamental spin-$(\frac{1}{2}, \frac{1}{2})$ XXZ R matrix is given by the 4×4 matrix

$$R(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}, \tag{2.2}$$

where η is the anisotropy parameter. It is a solution of the Yang–Baxter equation

$$R_{12}(u-v)\, R_{13}(u)\, R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u-v). \tag{2.3}$$

(See, e.g., refs. 17 and 18.) The fused spin-$(j, \frac{1}{2})$ R matrix ($j = \frac{1}{2}, 1, \frac{3}{2},\ldots$) is given by[17, 19]

$$\begin{aligned} &R_{\langle 1\cdots 2j\rangle\, 2j+1}(u) \\ &\quad = P^{+}_{1\cdots 2j} R_{1,\, 2j+1}(u)\, R_{2,\, 2j+1}(u+\eta) \cdots R_{2j,\, 2j+1}(u+(2j-1)\,\eta)\, P^{+}_{1\cdots 2j}. \end{aligned} \tag{2.4}$$

The (undeformed) projectors are defined by

$$P^{\pm}_{1\cdots m} = \frac{1}{m!} \sum_{\sigma} (\pm 1)^{\sigma}\, \mathscr{P}_{\sigma}, \tag{2.5}$$

where the sum is over all permutations $\sigma = (\sigma_1,\ldots, \sigma_m)$ of $(1,\ldots, m)$, and $\mathscr{P}_{\sigma}$ is the permutation operator in the space $\bigotimes_{k=1}^{m} \mathscr{C}^2$. For instance,

$$\begin{aligned} P^{+}_{12} &= \tfrac{1}{2}(\mathbb{I} + \mathscr{P}_{12}), \\ P^{+}_{123} &= \tfrac{1}{6}(\mathbb{I} + \mathscr{P}_{23}\mathscr{P}_{12} + \mathscr{P}_{12}\mathscr{P}_{23} + \mathscr{P}_{12} + \mathscr{P}_{23} + \mathscr{P}_{13}), \end{aligned} \tag{2.6}$$

where $\mathbb{I}$ is the identity matrix.

The closed-chain transfer matrix $t^{(j)}(u)$, which is constructed using a spin-j auxiliary space, is defined by

$$t^{(j)}(u) = \mathrm{tr}_{1\cdots 2j}\, T_{\langle 1\cdots 2j\rangle}(u), \tag{2.7}$$

where the fused monodromy matrix is defined by

$$T_{\langle 1\cdots 2j\rangle}(u) = R_{\langle 1\cdots 2j\rangle\, N}(u) \cdots R_{\langle 1\cdots 2j\rangle\, 1}(u), \tag{2.8}$$

and N corresponds to the number of spins of the chain. One can show that

$$T_{\langle 1\cdots 2j\rangle}(u) = P^{+}_{1\cdots 2j} T_1(u)\, T_2(u+\eta) \cdots T_{2j}(u+(2j-1)\,\eta)\, P^{+}_{1\cdots 2j}. \tag{2.9}$$

These transfer matrices constitute commutative families

$$[t^{(j)}(u), t^{(k)}(v)] = 0. \tag{2.10}$$

The fundamental transfer matrix $t(u) \equiv t^{(\frac{1}{2})}(u)$ contains the Hamiltonian (2.1),

$$H \propto \frac{\partial}{\partial u} \log t(u)\Big|_{u=0} + \text{const.}, \tag{2.11}$$

and has the periodicity property

$$t(u+i\pi) = (-1)^N\, t(u). \tag{2.12}$$

The fusion hierarchy for the XXZ chain is given by[17, 21]

$$t^{(j)}(u)\, t^{(\frac{1}{2})}(u+2j\eta) = \delta(u+(2j-1)\,\eta)\, t^{(j-\frac{1}{2})}(u) + t^{(j+\frac{1}{2})}(u), \qquad j=\tfrac{1}{2}, 1, \tfrac{3}{2},..., \tag{2.13}$$

where $t^{(0)}(u) = \mathbb{I}$, and the quantum determinant[17, 20] $\delta(u)$ is given by

$$\delta(u) = \operatorname{tr}_{12} P_{12}^{-} T_1(u)\, T_2(u+\eta) = (-\zeta(u+\eta))^N, \tag{2.14}$$

where

$$\zeta(u) = -\sinh(u+\eta)\,\sinh(u-\eta). \tag{2.15}$$

The key fact in deriving the functional relations is that for anisotropy values

$$\eta = \frac{i\pi}{p+1}, \qquad p = 1, 2,..., \tag{2.16}$$

the fused transfer matrices satisfy the identity

$$t^{(\frac{p+1}{2})}(u) = (-1)^N\, \delta(u-\eta) [t^{(\frac{p-1}{2})}(u+\eta) + (1+(-1)^N)\, v(u)^N\, F], \tag{2.17}$$

where

$$v(u) = -\frac{1}{\zeta(u)} \prod_{k=0}^{p} \sinh(u+k\eta) = -\frac{1}{\zeta(u)} \left(\frac{i}{2}\right)^p \sinh((p+1)\,u), \tag{2.18}$$

and

$$F = \prod_{k=1}^{N} \sigma_k^z. \tag{2.19}$$

The remarkable result (2.17), to which we refer as the "truncation identity," follows directly from Eq. (4.13) in ref. 14, which relies on the quantum-group construction[22] of higher-spin R matrices.

The fact that the spin-$\frac{p+1}{2}$ transfer matrix can be expressed in terms of a lower-spin transfer matrix leads to the truncation of the fusion hierarchy, which in turn leads to a $(p+1)$-order functional relation for the fundamental transfer matrix. For instance, for the case $p=2$, Eqs. (2.13) and (2.17) lead to the third-order functional relation

$$\begin{aligned} &t(u)\, t(u+\eta)\, t(u+2\eta) - \delta(u)\, t(u+2\eta) - \delta(u+\eta)\, t(u) \\ &\quad - (-1)^N\, \delta(u-\eta)\, t(u+\eta) - (1+(-1)^N)\, \delta(u-\eta)\, v(u)^N\, F = 0. \end{aligned} \tag{2.20}$$

Similar higher-order functional relations have been obtained for RSOS models[2, 23, 10] and for the 8-vertex model.[24] We emphasize that, contrary to the commonly-held misconception (see, e.g., ref. 25), the fusion hierarchies of vertex models *do* truncate, for the η values (2.16).

The commutativity relation (2.10) with $j=k=\frac{1}{2}$ and the fact $[F, t(u)]=0$ imply that $t(u)$ and F can be simultaneously diagonalized,

$$\begin{aligned} t(u)\, |\Lambda^{(\pm 1)}\rangle &= \Lambda^{(\pm 1)}(u)\, |\Lambda^{(\pm 1)}\rangle, \\ F\, |\Lambda^{(\pm 1)}\rangle &= \pm |\Lambda^{(\pm 1)}\rangle, \end{aligned} \tag{2.21}$$

where the eigenstates $|\Lambda^{(\pm 1)}\rangle$ are independent of u. Acting on these eigenstates with the functional relations, one obtains the corresponding relations for the eigenvalues.

2.2. Bethe Ansatz Solution

We now proceed to solve the functional relations for the eigenvalues of the fundamental transfer matrix. Following ref. 10, we observe that the functional relations for $p \geqslant 2$ can be represented in a compact form as the determinant of a $(p+1)\times(p+1)$ matrix:

$$\det \begin{pmatrix} \Lambda_0^{(F)} & -h_{-1} & 0 & 0 & \ldots & 0 & 0 & -Fh_0 \\ -h_1 & \Lambda_1^{(F)} & -h_0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -h_2 & \Lambda_2^{(F)} & -h_1 & \ldots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \ldots & -h_{p-1} & \Lambda_{p-1}^{(F)} & -h_{p-2} \\ -Fh_{p-1} & 0 & 0 & 0 & \ldots & 0 & -h_p & \Lambda_p^{(F)} \end{pmatrix} = 0, \tag{2.22}$$

where

$$h(u)=\sinh^{N}(u+\eta), \tag{2.23}$$

$h_k=h(u+\eta k)$, $\Lambda_k^{(F)}=\Lambda^{(F)}(u+\eta k)$, and $F=\pm 1$ now denotes the eigenvalue of the operator in (2.19). Let $(Q_0, Q_1,..., Q_p)$ be the null vector of the matrix in (2.22); i.e.,

$$\begin{aligned}
&\Lambda_0^{(F)} Q_0-h_{-1} Q_1-F h_0 Q_p=0, && \\
-&h_k Q_{k-1}+\Lambda_k^{(F)} Q_k-h_{k-1} Q_{k+1}=0, &\qquad & k=1,..., p-1, \\
-&F h_{p-1} Q_0-h_p Q_{p-1}+\Lambda_p^{(F)} Q_p=0. &&
\end{aligned} \tag{2.24}$$

We make the Ansatz $Q_k=Q(u+\eta k)$, where

$$Q(u)=\prod_{j=1}^{M} \sinh(u-u_j), \tag{2.25}$$

for some integer M. Equations (2.24) imply (using $Q_k=(-1)^M Q_{k+p+1}$) that the eigenvalues are given by

$$\Lambda^{(F)}(u)=h(u) \frac{Q(u-\eta)}{Q(u)}+h(u-\eta) \frac{Q(u+\eta)}{Q(u)}, \tag{2.26}$$

and

$$F=(-1)^M. \tag{2.27}$$

We verify that the result (2.26) is consistent with the periodicity condition (2.12). The requirement that $\Lambda^{(F)}(u)$ be analytic at $u=u_j$ yields the Bethe Ansatz equations

$$\frac{h(u_j)}{h(u_j-\eta)}=-\frac{Q(u_j+\eta)}{Q(u_j-\eta)}, \qquad j=1,..., M. \tag{2.28}$$

We recognize Eqs. (2.23), (2.25), (2.26), and (2.28) as the familiar Bethe Ansatz result for the eigenvalues of the transfer matrix of the closed XXZ chain. Although we have assumed that η has the values (2.16), these results are known to be true for general values of η. Note also that the approach that we have followed here does not explicitly rely on the existence of a pseudovacuum state.

A more thorough analysis would also include the diagonalization of $S^z = \frac{1}{2}\sum_{k=1}^{N} \sigma_k^z$ along with $t(u)$ and F. By considering the asymptotic behavior of $t(u)$ for $u \to \infty$, one can establish that the value of M is related to the S^z eigenvalue, namely, $M = \frac{N}{2} - S^z$. Since these matters are already well-understood, and since the open-chain problem (1.1) lacks this additional $U(1)$ symmetry, we do not pursue this issue further.

Finally, we point out that the result (2.27), which perhaps is less familiar, can nevertheless be readily obtained within the algebraic Bethe Ansatz approach.[5] Indeed, as is well known, the eigenstates of the transfer matrix are given by

$$|u_1,\ldots, u_M\rangle = B(u_1)\cdots B(u_M)\,|\Omega\rangle, \tag{2.29}$$

where $B(u)$ is a certain creation-like operator, and $|\Omega\rangle$ is the pseudo-vacuum state with all spins up. It is easy to show that $\{F, B(u)\} = 0$ and $F\,|\Omega\rangle = |\Omega\rangle$. Hence, the state $|u_1,\ldots, u_M\rangle$ has the eigenvalue $F = (-1)^M$.

3. THE OPEN CHAIN

We turn now to the open chain (1.1), which is our main concern. Our strategy is to try to generalize the analysis of the preceding section. Hence, in Section 3.1 we review the functional relations, and then in Section 3.2 we attempt to solve them.

3.1. Functional Relations

The fundamental spin-$\frac{1}{2}$ XXZ K^- matrix is given by the 2×2 matrix[11, 12]

$$K^-(u) = \begin{pmatrix} \sinh(\xi_- + u) & \kappa_- \sinh 2u \\ \kappa_- \sinh 2u & \sinh(\xi_- - u) \end{pmatrix}, \tag{3.1}$$

which evidently depends on two boundary parameters ξ_-, κ_-. It is a solution of the boundary Yang–Baxter equation[26]

$$R_{12}(u-v)\, K_1^-(u)\, R_{21}(u+v)\, K_2^-(v) = K_2^-(v)\, R_{12}(u+v)\, K_1^-(u)\, R_{21}(u-v). \tag{3.2}$$

The fundamental open-chain transfer matrix $t(u)$ is constructed, following Sklyanin's recipe,[6] from the matrix $R(u)$ (2.2), the matrix $K^-(u)$ (3.1), and the matrix $K^+(u)$ which is equal to $K^-(-u-\eta)$ with (ξ_-, κ_-)

replaced by (ξ_+, κ_+).[3] The Hamiltonian (1.1) is related to the first derivative of the transfer matrix,

$$\mathscr{H} = \frac{1}{4\sinh\xi_-\sinh\xi_+\sinh^{2N-1}\eta\cosh\eta}\left.\frac{\partial t(u)}{\partial u}\right|_{u=0} - \frac{\sinh^2\eta + N\cosh^2\eta}{2\cosh\eta}\mathbb{I}. \tag{3.3}$$

The transfer matrix has the periodicity property

$$t(u+i\pi) = t(u), \tag{3.4}$$

as well as crossing symmetry

$$t(-u-\eta) = t(u), \tag{3.5}$$

and the asymptotic behavior (for $\kappa_\pm \neq 0$)

$$t(u) \sim -\kappa_-\kappa_+ \frac{e^{u(2N+4)+\eta(N+2)}}{2^{2N+1}}\mathbb{I} + \cdots \qquad \text{for} \quad u \to \infty. \tag{3.6}$$

Functional relations for the open XXZ chain (1.1) have been proposed in ref. 14. These relations, which follow from the fusion hierarchy[27, 28] together with the truncation identity for the η values (2.16), are given by

$$\begin{aligned}
&\Lambda(u)\,\Lambda(u+\eta)\cdots\Lambda(u+p\eta) \\
&\quad -\delta(u-\eta)\,\Lambda(u+\eta)\,\Lambda(u+2\eta)\cdots\Lambda(u+(p-1)\,\eta) \\
&\quad -\delta(u)\,\Lambda(u+2\eta)\,\Lambda(u+3\eta)\cdots\Lambda(u+p\eta) \\
&\quad -\delta(u+\eta)\,\Lambda(u)\,\Lambda(u+3\eta)\,\Lambda(u+4\eta)\cdots\Lambda(u+p\eta) \\
&\quad -\delta(u+2\eta)\,\Lambda(u)\,\Lambda(u+\eta)\,\Lambda(u+4\eta)\cdots\Lambda(u+p\eta) - \cdots \\
&\quad -\delta(u+(p-1)\,\eta)\,\Lambda(u)\,\Lambda(u+\eta)\cdots\Lambda(u+(p-2)\,\eta) + \cdots \\
&= f(u),
\end{aligned} \tag{3.7}$$

where $\Lambda(u)$ is the eigenvalue of the fundamental open-chain transfer matrix $t(u)$. Furthermore, the function $\delta(u)$ is now defined by

$$\delta(u) = \frac{\Delta(u)}{\zeta(2u+2\eta)}, \tag{3.8}$$

[3] Further details about the construction of this transfer matrix can be found in ref. 14.

where the quantum determinant $\Delta(u)$ is given by

$$\begin{aligned}\Delta(u) = &-[\sinh(u+\eta+\xi_-) \sinh(u+\eta-\xi_-)+\kappa_-^2 \sinh^2(2u+2\eta)]\\ &\times [\sinh(u+\eta+\xi_+) \sinh(u+\eta-\xi_+)+\kappa_+^2 \sinh^2(2u+2\eta)]\\ &\times \sinh 2u \sinh(2u+4\eta)\, \zeta(u+\eta)^{2N},\end{aligned} \tag{3.9}$$

and $\zeta(u)$ is defined in Eq. (2.15). Moreover, the function $f(u)$ is given by[4]

$$\begin{aligned}f(u) = &\frac{(-1)^{p(N+1)}}{2^{2p(N+1)}} \sinh^{2N}((p+1)\, u) \frac{\cosh^2((p+1)\, u+\frac{i\pi}{2}\, \epsilon)}{\cosh^2((p+1)\, u)}\\ &\times \{n(u;\, \xi_-, \kappa_-)\, n(u; -\xi_+, \kappa_+)+n(u; -\xi_-, \kappa_-)\, n(u;\, \xi_+, \kappa_+)\\ &+2(-1)^N\, (-\kappa_-\kappa_+)^{p+1} \sinh^2(2(p+1)\, u)\},\end{aligned} \tag{3.10}$$

where $\epsilon = 2\mathrm{frac}(p/2)$ equals 0 if p is even, and equals 1 if p is odd; and the function $n(u;\, \xi, \kappa)$ is defined by

$$n(u;\, \xi, \kappa) = \sinh((p+1)(\xi+u))+\sum_{l=1}^{\left[\frac{p+1}{2}\right]} c_{p,\,l}\kappa^{2l} \sinh((p+1)\, u+(p+1-2l)\, \xi), \tag{3.11}$$

with

$$c_{p,\,l} = \frac{(p+1)}{l!} \prod_{k=0}^{l-2} (p-l-k).$$

For instance, for the case $p=3$, the functional relation is given by[5]

$$\begin{aligned}&\Lambda(u)\, \Lambda(u+\eta)\, \Lambda(u+2\eta)\, \Lambda(u+3\eta)-\delta(u-\eta)\, \Lambda(u+\eta)\, \Lambda(u+2\eta)\\ &\quad -\delta(u)\, \Lambda(u+2\eta)\, \Lambda(u+3\eta)-\delta(u+\eta)\, \Lambda(u)\, \Lambda(u+3\eta)\\ &\quad -\delta(u+2\eta)\, \Lambda(u)\, \Lambda(u+\eta)+\delta(u)\, \delta(u+2\eta)+\delta(u-\eta)\, \delta(u+\eta) = f(u).\end{aligned} \tag{3.12}$$

[4] In terms of the three functions f_0, f_1, f_3 used in ref. 14, the functions $\delta(u)$ and $f(u)$ are given by

$$\delta(u) = \frac{f_1(u+\eta)}{f_0(u)}, \qquad f(u) = \frac{f_3(u)}{f_0(u)}.$$

[5] The last two terms of the left-hand-side were accidentally omitted in Eq. (5.2) of ref. 14.

3.2. Bethe Ansatz Solution

For general values of the boundary parameters $\kappa_{\pm}$, $\xi_{\pm}$, we have not yet succeeded to find a determinant representation analogous to (2.22) of the functional relations (3.7). Nevertheless, for the following two special cases of the boundary parameters, we have found such representations.

3.2.1. The Diagonal Case $\kappa_{\pm}=0$

The Bethe Ansatz solution for the diagonal case $\kappa_{\pm}=0$ is already known.[3,6] Nevertheless, it is instructive to see how this solution emerges from the functional relations. Indeed, when $\kappa_{\pm}=0$, the functional relations for $p \geqslant 2$ can be represented as

$$\det \begin{pmatrix} \Lambda_0 & -h'_{-1} & 0 & 0 & \dots & 0 & 0 & -h_0 \\ -h_1 & \Lambda_1 & -h'_0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -h_2 & \Lambda_2 & -h'_1 & \dots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \dots & -h_{p-1} & \Lambda_{p-1} & -h'_{p-2} \\ -h'_{p-1} & 0 & 0 & 0 & \dots & 0 & -h_p & \Lambda_p \end{pmatrix} = 0, \tag{3.13}$$

where

$$h(u) = -\sinh^{2N}(u+\eta)\,\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\,\sinh(u+\xi_-)\,\sinh(u-\xi_+), \tag{3.14}$$

$$h'(u) = h(-u-2\eta), \tag{3.15}$$

and $h_k = h(u+\eta k)$, $h'_k = h'(u+\eta k)$, $\Lambda_k = \Lambda(u+\eta k)$. We let $(Q_0, Q_1,\dots, Q_p)$ be the null vector of the matrix in (3.13); i.e.,

$$\begin{aligned} \Lambda_0 Q_0 - h'_{-1} Q_1 - h_0 Q_p &= 0, \\ -h_k Q_{k-1} + \Lambda_k Q_k - h'_{k-1} Q_{k+1} &= 0, \qquad k = 1,\dots, p-1, \\ -h'_{p-1} Q_0 - h_p Q_{p-1} + \Lambda_p Q_p &= 0. \end{aligned} \tag{3.16}$$

We make the Ansatz $Q_k = Q(u+\eta k)$, where

$$Q(u) = \prod_{j=1}^{M} \sinh(u-u_j)\,\sinh(u+u_j+\eta), \tag{3.17}$$

which has the crossing symmetry $Q(u)=Q(-u-\eta)$. Equations (3.16) and (3.15) imply that the eigenvalues are given by

$$\Lambda(u)=h(u)\,\frac{Q(u-\eta)}{Q(u)}+h(-u-\eta)\,\frac{Q(u+\eta)}{Q(u)}. \tag{3.18}$$

We verify that this result is consistent with both the periodicity (3.4) and crossing (3.5) properties of the transfer matrix. The requirement that $\Lambda(u)$ be analytic at $u=u_j$ yields the Bethe Ansatz equations

$$\frac{h(u_j)}{h(-u_j-\eta)}=-\frac{Q(u_j+\eta)}{Q(u_j-\eta)}, \qquad j=1,\dots,M. \tag{3.19}$$

The results (3.14), (3.17)–(3.19) for the transfer-matrix eigenvalues and Bethe Ansatz equations agree with those of Alcaraz *et al.*[3] and Sklyanin.[6] Although we have assumed that η has the values (2.16), these results are true for general values of η. As in the case of the closed chain, one can establish that $M=\frac{N}{2}-S^z$ by considering the asymptotic behavior of $t(u)$ for $u\to\infty$.

3.2.2. The Nondiagonal Case $\kappa_+=\kappa_-\equiv\kappa$, $\xi_+=\xi_-\equiv\xi$, $N=odd$

Finally, we consider the nondiagonal case $\kappa_+=\kappa_-\equiv\kappa\neq 0$, $\xi_+=\xi_-\equiv\xi$, $N=$ odd. For this case, the functional relations again have the determinant representation (3.13), with

$$h(u)=-\sinh^{2N}(u+\eta)\,\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\,(\sinh(u+\xi)\,\sinh(u-\xi)+\kappa^2\sinh^2 2u). \tag{3.20}$$

It follows that the transfer-matrix eigenvalues and Bethe Ansatz equations are again given by (3.17)–(3.19). However, unlike the two cases considered earlier which have a $U(1)$ symmetry, here the value of M is fixed. Indeed, the asymptotic behavior (3.6) implies that

$$M=\tfrac{1}{2}(N-1). \tag{3.21}$$

We expect that, as in the previous cases, these results hold for general values of η.

4. DISCUSSION

We have seen that an approach used by Bazhanov and Reshetikhin[10] to solve RSOS models, which is based on a model's functional relations, is also an effective means of solving vertex models. Indeed, we have used this method to recover the known Bethe Ansatz solutions of both the closed and open XXZ chains with $U(1)$ symmetry. Moreover, since this method does not rely on the existence of a pseudovacuum state, we have also been able to use this method to solve the special nondiagonal case (1.2) of the open chain.

Although we have focused here on vertex models associated with sl_2, it is clear that the same approach should be applicable to vertex models associated with higher-rank algebras.

Having found a model's functional relations, a crucial step in this method is to reformulate the functional relations in determinant form. We have not yet succeeded to carry out this step for general values of the boundary parameters of the open XXZ chain (1.1). It would clearly be useful to find necessary and sufficient conditions for the existence of a determinant representation of the functional relations, as well as a systematic procedure for its construction. We hope to be able to report on these matters in a future publication.

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